REDUCTION FORMULAE PROBLEMS

Additional results :		
I.	$\int_0^{\pi} \sin^n x\, dx = 2\int_0^{\pi/2} \sin^n x\, dx$	For all integral values of n.
II.	$\int_0^{\pi} \cos^n x\, dx = 2\int_0^{\pi/2} cos^n x\, dx$ $= \quad 0$	If n is an even integer. If n is an odd integer.
III.	$\int_0^{2\pi} \sin^n x\, dx = 4\int_0^{\pi/2} \sin^n x\, dx$ $= 0$	If n is an even integer. If n is an odd integer.
IV.	$\int_0^{2\pi} \cos^n x\, dx = 4\int_0^{\pi/2} \cos^n x\, dx$ $= 0$	If n is an even integer. If n is an odd integer.
V.	$\int_0^{\pi/2} \sin^p x \;\; \cos x\, dx = \int_0^{\pi/2} \sin x \;\cos^p x\, dx$ $= \dfrac{1}{p+1}$	
VI.	$\int_0^{\pi} \sin^m x \;\cos^n x\, dx = 2\int_0^{\pi/2} \sin^m x \;\cos^n x\, dx$ $= \quad 0$	If n is even, m is even or odd If n is odd, m is even or odd

VII.	$\int_0^{2\pi} \sin^m x \ \cos^n x \, dx = 4\int_0^{\pi/2} \sin^m x \ \cos^n x \, dx$	If m and n both are even
	$= \quad 0$	Otherwise

Ex. 1 : $\displaystyle\int_0^1 \frac{x^7}{\sqrt{1-x^2}}\, dx$

Solution :

Put $x = \sin\theta \qquad dx = \cos\theta\, d\theta$

When $x = 0, \quad \theta = 0$

$x = 1, \quad \theta = \pi/2$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\sin^7\theta}{\cos\theta}\cdot\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin 7\,\theta\, d\theta$$

$$= \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot 1$$

$$= \frac{16}{35}$$

Ex. 2 : $\displaystyle\int_0^1 \frac{x^2\sqrt{4-x^2}}{\sqrt{1-x^2}}\, dx$

Solution :

Put $x = \sin\theta \qquad dx = \cos\theta\, d\theta$

$\theta = 0$ to $\pi/2$

$$I = \int_0^{\pi/2} \frac{\sin^2\theta\,(4-\sin^2\theta)}{\cos\theta}\cos\theta\, d\theta$$

$$= 4\int_0^{\pi/2} \sin^2\theta\, d\theta - \int_0^{\pi/2} \sin^4 4\, d\theta$$

$$= 4\left(\frac{1}{2}\cdot\frac{\pi}{2}\right) - \left(\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)$$

$$= \pi - \frac{3\pi}{16}$$

Ex. 3: $\displaystyle\int_0^3 \frac{x^{3/2}}{\sqrt{3-x^{1/2}}}\, dx$

Solution :

Put $x = 3\sin^2\theta \qquad dx = 6\sin\theta\cos\theta\, d\theta$

$\theta = 0 \quad \text{to} \quad \pi/2$

$$I = \int_0^{\pi/2} \frac{(3\sin^2\theta)^{3/2}}{\left(3 - 3\sin^2\theta\right)^{1/2}} \cdot 6\sin\theta\cos\theta\, d\theta$$

$$I = \int_0^{\pi/2} \frac{3^{3/2}\ \sin^3\theta}{3^{1/2}\cos\theta}\ 6\sin\theta\cos\theta\, d\theta$$

$$= 18\int_0^{\pi/2} \sin^4 a\, d\theta$$

$$= 18^9 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\mathbf{I = \frac{27\pi}{8}}$$

Ex. 4 : Find the reduction formula for $\int_0^{\frac{\pi}{4}} \sin^n x\, dx$ and hence evaluate $\int_0^{\frac{\pi}{4}} \sin^6 x\, dx$

Solution :

Let $I_n = \int_0^{\frac{\pi}{4}} \sin^n x\, dx$

$$= \int_0^{\frac{\pi}{4}} \sin^{n-1} x \sin x\, dx$$

Integrating by parts,

$$= \sin^{n-1} x \int \sin x\, dx - \int_0^{\frac{\pi}{4}} \left(\frac{d}{dx}\sin^{n-1} x\right)\left(\int \sin x\, dx\right) dx$$

$$= [\sin^{n-1} x\,(-\cos x)]_0^{\pi/4} - \int_0^{\frac{\pi}{4}} [(n-1)\sin^{n-2} x\,(\cos x)]\,(-\cos x)\, dx$$

$$= \left[\left(\sin\frac{\pi}{4}\right)^{n-1}\ \left(\cos\frac{\pi}{4}\right) - 0\right] + (n-1)\int_0^{\frac{\pi}{4}} \sin^{n-2} x\cos^2 x\, dx$$

$$= -\left[\left(\frac{1}{\sqrt{2}}\right)^{n-1}\ \left(\frac{1}{\sqrt{2}}\right)\right] + (n-1)\int_0^{\frac{\pi}{4}} \sin^{n-2} x\,(1 - \sin^2 x)\, dx$$

$$= -\left(\frac{1}{\sqrt{2}}\right)^n + (n-1)\left[\int_0^{\frac{\pi}{4}} \sin^{n-2} x\, dx - \int_0^{\frac{\pi}{4}} \sin^n x\, dx\right]$$

$$I_n = -\left(\frac{1}{\sqrt{2}}\right)^n + (n-1)\,[I_{n-2} - I_n]$$

$$I_n + (n-1)\, I_n = -\left(\frac{1}{\sqrt{2}}\right)^n + (n-1)\, I_{n-2}$$

$$n\, I_n = -\left(\frac{1}{\sqrt{2}}\right)^n + (n-1)\, I_{n-2}$$

$$I_n = -\frac{1}{n} - \left(\frac{1}{\sqrt{2}}\right)^n + \frac{n-1}{n}\, I_{n-2} \qquad \ldots(1)$$

Now, $\int_0^{\frac{\pi}{4}} \sin^6 x\, dx = I_6$

$\therefore$ Put n = 6 in equation (1)

$$I_6 = -\frac{1}{6}\left(\frac{1}{\sqrt{2}}\right)^6 + \frac{5}{6}\, I_4$$

$$I_6 = -\frac{1}{48} + \frac{5}{6}\, I_4 \qquad \ldots(2)$$

Put n = 4 in equation (1)

$$I_4 = -\frac{1}{4}\left(\frac{1}{\sqrt{2}}\right)^4 + \frac{3}{4}\, I_2$$

$$I_4 = -\frac{1}{16} + \frac{3}{4}\, I_2 \qquad \ldots(3)$$

Put n = 2 in equation (1)

$$I_2 = -\frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}\, I_0$$

$$I_2 = -\frac{1}{4} + \frac{1}{2}\, I_0 \qquad \ldots(4)$$

But $I_0 = \int_0^{\frac{\pi}{4}} \sin^0 x\, dx = \int_0^{\frac{\pi}{4}} 1\, dx = \frac{\pi}{4}$

$\therefore$ Equation (4) $\Rightarrow$ $I_2 = -\frac{1}{4} + \frac{1}{2}\left(\frac{\pi}{4}\right)$

$$I_2 = -\frac{1}{4} + \frac{\pi}{4}$$

Equation (3) $\Rightarrow$ $I_4 = -\frac{1}{16} + \frac{3}{4}\left(-\frac{1}{4} + \frac{\pi}{4}\right)$

$$= -\frac{1}{16} - \frac{3}{16} + \frac{3\pi}{32}$$

$$I_4 = -\frac{1}{4} + \frac{3\pi}{32}$$

Equation (2) $\Rightarrow$ $I_6 = -\frac{1}{48} + \frac{5}{6}\left(-\frac{1}{4} + \frac{3\pi}{32}\right)$

$$= -\frac{1}{48} - \frac{5}{24} + \frac{15\pi}{6 \times 32}$$

$$= \frac{-1-10}{48} + \frac{5\pi}{64}$$

$$I_6 = -\frac{11}{48} + \frac{5\pi}{64}$$

$$\therefore \int_0^{\frac{\pi}{4}} \sin^6 x \, dx = -\frac{11}{48} + \frac{5\pi}{64}$$

Ex.: Find R.F. for $\int \tan^{n} x\, dx$

Solution:

$$Let\ U_n = \int \tan^{n} x\, dx$$

$$= \int \tan^{n-2} x \tan^2 x\, dx = \int \tan^{n-2} x\,(sec^2 x - 1)dx$$

$$= \int \tan^{n-2} x\, sec^2 x\, dx - \int \tan^{n-2} x\, dx$$

$$= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx$$

$$\therefore U_n = \frac{\tan^{n-1} x}{n-1} - U_{n-2}$$

Ex: $If\ I_{m,n} = \int \cos^{m} x \sin nx\, dx\,.\ then\ show\ that\ (m+n)I_{m,n} = -\cos^{m} x \cos nx + mI_{m-1,n-1}$

Solution:

We have,

$$I_{m,n} = \int \cos^{m} x \sin nx\, dx$$

Using by parts

$$I_{m,n} = \cos^{m} x \int \sin nx\, dx - \int \frac{d}{dx}(\cos^{m} x)\left(\int \sin nx\, dx\right) dx$$

$$= -\frac{\cos nx \cos^{m} x}{n} - \int -m\cos^{m} x \sin x \left(-\frac{\cos nx}{n}\right) dx$$

$$= -\frac{\cos nx \cos^{m} x}{n} - \frac{m}{n}\int \cos^{m} x \sin x \cos nx\, dx \qquad \ldots(1)$$

Consider

$$\sin(n-1)x = \sin nx \cos x - \cos nx \sin x$$

$$\therefore \cos nx \sin x = \sin nx \cos x - \sin(n-1)x$$

∴ equation (1) becomes,

$$I_{m,n} = -\frac{\cos nx \cos^{m} x}{n} - \frac{m}{n}\int \cos^{m} x\,[\sin nx \cos x - \sin(n-1)x]dx$$

$$I_{m,n} = -\frac{\cos nx \cos^{m} x}{n} - \frac{m}{n}\int \cos^{m-1} x \sin nx \cos x\, dx + \frac{m}{n}\int \cos^{m-1} x \sin(n-1)x\, dx$$

$$I_{m,n} = -\frac{\cos nx \cos^{m} x}{n} - \frac{m}{n}\int \cos^{m} x \sin nx\, dx + \frac{m}{n}\int \cos^{m-1} x \sin(n-1)x\, dx$$

$$I_{m,n} = -\frac{\cos nx \cos^{m} x}{n} - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,n-1}$$

$$i.e. \qquad (m+n)I_{m,n} = -\cos^{\mathrm{m}} x \cos nx + mI_{m-1,n-1}$$

$$I_{m,n} = -\cos^{\mathrm{m}} x \cos nx + mI_{m-1,n-1}$$

GAMMA FUNCTION AND ITS PROPERTIES.

Definition : Gamma function

$$\left\lceil \mathbf{n} \right. = \int_{\mathbf{0}}^{\infty} \boldsymbol{e^{-x} x^{n-1} dx}\ (\boldsymbol{n} > 0)$$

Properties :	
1.	$\left\lceil \mathbf{n} \right. = 2\int_{\mathbf{0}}^{\infty} \boldsymbol{e^{-x2} x^{2n-1} dx}$
2.	$\left\lceil \mathbf{1} \right. = \mathbf{1}$
3.	$\left\lceil \frac{\mathbf{1}}{\mathbf{2}} \right. = \sqrt{\boldsymbol{\pi}}$
4.	$\left\lceil \boldsymbol{n+1} \right. = \boldsymbol{n} \left\lceil \boldsymbol{n} \right.$

Ex.: If $I_n = \int_0^{\pi/4} \frac{\sin(2n-1)x}{\sin x} dx$ then prove that$n(I_{n+1} - I_n) = \sin \frac{n\pi}{2}$. Hence find I_3

Solution:

From I_n we can write

$$I_{n+1} = \int_0^{\pi/4} \frac{\sin(2n+1)x}{\sin x} dx$$

$$\therefore I_{n+1} - I_n = \int_0^{\frac{\pi}{4}} \frac{\sin(2n+1)x - \sin(2n-1)x}{\sin x} dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{2\cos 2nx \,.\sin x}{\sin x}\,dx = 2\int_0^{\frac{\pi}{4}} \cos 2nx\,dx = 2\left[\frac{\sin 2nx}{2n}\right]_0^{\pi/4} = \frac{1}{n}\sin\frac{n\pi}{2}$$

To find I_3,

Put $n = 2, I_3 - I_2 = 0$

Put n=1, $I_2 - I_1 = 1 \;\therefore I_2 = 1 + I_1$

But $I_1 = \int_0^{\frac{\pi}{4}} 1.\,dx = \frac{\pi}{4}$

$$\therefore I_3 = 1 + \frac{\pi}{4}$$

Combination of Algebraic and Trigonometric Functions

Ex.: If $I_n = \int_0^{\pi/2} x\sin^{\mathrm{n}} x\,dx$ show that $I_n = \frac{n-1}{n} I_{n-2} + \frac{1}{n^2}$

Solution: We have,

$$I_n = \int_0^{\pi/2} x\sin^{\mathrm{n}} x\,dx = I_n = \int_0^{\pi/2} (x\sin x)\sin^{\mathrm{n-1}} x\,dx$$

Using the rule of by parts

$$I_n = \left[\sin^{\mathrm{n-1}} x \int x\sin x\,dx\right]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} \frac{d}{dx}(\sin^{\mathrm{n-1}} x)\left(\int x\sin x\,dx\right)dx$$

$$I_n = [\sin^{\mathrm{n-1}} x\,(-xcos\,x + \sin x)]_0^{\pi/2}$$
$$- \int_0^{\frac{\pi}{2}} (n-1)(\sin^{\mathrm{n-2}} x\cos x)\,((-xcos\,x + \sin x))dx$$

$$I_n = 1 + (n-1)\int_0^{\frac{\pi}{2}} x(\sin^{\mathrm{n-2}} x\cos^2 x)\,dx - (n-1)\int_0^{\pi/2} \sin^{\mathrm{n-1}} x\cos x\,dx$$

$$I_n = 1 + (n-1)\int_0^{\frac{\pi}{2}} x(\sin^{\mathrm{n-2}} x\,(1 - \sin^2 x))\,dx$$

$$-(n-1)\int_0^{\pi/2} \sin^{\mathrm{n-1}} x\cos x\,dx$$

$$I_n = 1 + (n-1)\int_0^{\frac{\pi}{2}} x(\sin^{\mathrm{n}-2} x\,(1-\sin^2 x))\,dx$$

$$-(n-1)\int_0^{\pi/2} \sin^{\mathrm{n}-1} x \cos x\,dx$$

$$\therefore I_n = 1 + (n-1)I_{n-2} - (n-1)I_n - \frac{n-1}{n}$$

Which on simplification gives

$$I_n = \frac{n-1}{n} I_{n-2} + \frac{1}{n^2}$$

R.F. For Algebraic Functions

Ex: If $I_n = \int \frac{x^n}{(a^2+x^2)^{3/2}}\,dx$ then show that

$$(n-2)I_n = \frac{x^{n-1}}{(a^2+x^2)^{\frac{1}{2}}} - (n-1)a^2 I_{n-2}$$

Solution: Consider

$I_n = \int x^{n-1} \frac{x}{(a^2+x^2)^{3/2}}\,dx$ and apply by parts we get

$$I_n = x^{n-1}\int \frac{x}{(a^2+x^2)^{3/2}}\,dx - \int \frac{d}{dx}(x^{n-1})\left(\int \frac{x}{(a^2+x^2)^{\frac{3}{2}}}\,dx\right)dx$$

$$I_n = -x^{n-1}\frac{1}{(a^2+x^2)^{\frac{1}{2}}} + (n-1)\int x^{n-2}\frac{1}{(a^2+x^2)^{\frac{1}{2}}}\,dx$$

Multiplying and dividing integral by(a^2+x^2)

$$I_n = -x^{n-1}\frac{1}{(a^2+x^2)^{\frac{1}{2}}} + (n-1)\int x^{n-2}\frac{(a^2+x^2)}{(a^2+x^2)^{\frac{3}{2}}}\,dx$$

$$I_n = -x^{n-1}\frac{1}{(a^2+x^2)^{\frac{1}{2}}} + (n-1)\int \frac{a^2x^{n-2}}{(a^2+x^2)^{\frac{3}{2}}}dx$$

$$+(n-1)\int \frac{x^n}{(a^2+x^2)^{\frac{3}{2}}}dx$$

$$I_n = -x^{n-1}\frac{1}{(a^2+x^2)^{\frac{1}{2}}} + (n-1)a^2I_{n-2} + (n-1)I_n$$

On simplification we get

$$(n-2)I_n = \frac{x^{n-1}}{(a^2+x^2)^{\frac{1}{2}}} - (n-1)a^2I_{n-2}$$

GAMMA FUNCTION, BETA FUNCTION

Definition:

The definite improper integral $\int_0^\infty t^{n-1}e^{-t}dt\,,(n>0)$ is denoted by the symbol Γ(n) and called as Gamma n

Thus

Γ(n)= $\int_0^\infty t^{n-1}e^{-t}dt\,,(n>0)$

Properties of Gamma Functions:

1. $\Gamma(1) = 1$

Proof: $\Gamma(n)$= $\int_0^\infty t^{n-1}e^{-t}dt$

Put n=1

$$\Gamma(1)=\int_0^\infty t^{1-1}e^{-t}dt=\int_0^\infty e^{-t}dt=1$$

2. Reduction formula for Gamma Function: $\Gamma(n+1)=n\Gamma(n)$

Proof: $\Gamma(n+1)= \int_0^\infty t^n e^{-t}dt$

Using by parts

$$\Gamma(n+1)=\left[t^n\int e^{-t}\,dt\right]_0^\infty-\int_0^\infty \frac{d}{dt}t^n\left(\int e^{-t}dt\right)dt$$

$$\Gamma(n+1)=\left[t^n\left(\frac{e^{-t}}{-1}\right)\right]_0^\infty-\int_0^\infty nt^{n-1}\left(\frac{e^{-t}}{-1}\right)dt$$

$$\Gamma(n+1)=0+n\int_0^\infty t^{n-1}e^{-t}dt$$

$$\therefore \Gamma(n+1)=n\Gamma(n)$$

3. If n is positive integer then $\Gamma(n)=(n-1)!$
4. $\Gamma(0)=\infty$
5. $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$
6. $\int_0^\infty t^{n-1}e^{-kt}dt=\frac{\Gamma(n)}{k^n}$

Proof: Consider, $I=\int_0^\infty t^{n-1}e^{-kt}dt$

Put $kt=x \Longrightarrow t=\frac{x}{k}\ \therefore dt=\frac{dx}{k}$

t	0	∞
x	0	∞

$$\therefore I=\int_0^\infty e^{-x}\left(\frac{x}{k}\right)^{n-1}\frac{dx}{k}$$

$$\therefore I = \frac{1}{k^n}\int_0^\infty e^{-x}x^{n-1}dx = \frac{\Gamma(n)}{k^n}$$

7. $\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi}$

e.g.: $\Gamma(1/3)\Gamma(2/3) = \frac{\pi}{\sin\frac{\pi}{3}} = \frac{\pi}{\sqrt{3}/2} = \frac{2\pi}{\sqrt{3}}$

Ex.1: Evaluate $\int_0^\infty x^{3/2}\, e^{-x}\, dx$

Solution: $\int_0^\infty x^{3/2}\, e^{-x}\, dx = \int_0^\infty x^{5/2-1}\, e^{-x}\, dx$

$= \Gamma(5/2)$

$= \Gamma(3/2+1)$

$= 3/2\ \Gamma(3/2)$

$= 3/2 \cdot ½\ \Gamma(½)$

$= 3/2 \cdot ½ \cdot \sqrt{\pi}$

$= ¾ \sqrt{\pi}$

Ex. 2: Find $\Gamma(-½)$

Solution: $(-½) + 1 = ½$

$\Gamma(-1/2) = \Gamma(-½ + 1) / (-½)$

$= -2\ \Gamma(1/2)$

$= -2 \sqrt{\pi}$

Some Standard integrals to be reduced to Gamma Function

1. $I = \int_0^\infty x^{n-1}e^{-ax^b}\, dx$

Solution: Consider $I = \int_0^\infty x^{n-1}e^{-ax^b}\, dx$

Put, $ax^b = t \Longrightarrow x = \frac{t^{\frac{1}{b}}}{a^{\frac{1}{b}}}$

Thus, $dx = \frac{1}{b}\frac{1}{a^{\frac{1}{b}}}t^{\frac{1}{b}-1}dt,$

x	0	∞
t	0	∞

$$\therefore I = \int_0^\infty \frac{t^{\frac{n-1}{b}}}{a^{\frac{n-1}{b}}} e^{-t}\frac{1}{b}\frac{1}{a^{\frac{1}{b}}}t^{\frac{1}{b}-1}dt$$

$$\therefore I = \frac{1}{b}\frac{1}{a^{\frac{n}{b}}}\int_0^\infty \frac{t^{\frac{n}{b}-1}}{a^{\frac{n-1}{b}}}e^{-t}dt$$

$$i.e. I = \frac{1}{b}\frac{1}{a^{\frac{n}{b}}}\ \Gamma(n/b)$$

2. $I = \int_0^\infty x^{n-1}a^{-bx^m}dx$

Solution:

Consider $I = \int_0^\infty x^{n-1}a^{-bx^m}dx$

Put $a^{-bx^m} = e^{-t} \Longrightarrow x = \frac{t^{\frac{1}{m}}}{(b\log a)^{\frac{1}{m}}}$

$$\therefore dx = \frac{1}{m}\frac{1}{b^{\frac{1}{m}}}\frac{1}{(\log a)^{\frac{1}{m}}}t^{\frac{1}{m}-1}dt$$

$$\therefore I = \int_0^\infty \left[\frac{t^{\frac{1}{m}}}{(b\log a)^{\frac{1}{m}}}\right]^{n-1} e^{-t}\frac{1}{m}\frac{1}{b^{\frac{1}{m}}}\frac{1}{(\log a)^{\frac{1}{m}}}t^{\frac{1}{m}-1}dt$$

$$I = \frac{1}{mb^{\frac{n}{m}}(\log a)^{\frac{n}{m}}}\int_0^\infty t^{\frac{n}{m}-1}e^{-t}dt = \frac{1}{mb^{\frac{n}{m}}(\log a)^{\frac{n}{m}}}\Gamma(n/m)$$

3. $I = \int_0^1 x^m(\log x)^n dx$

Solution:

Put $x = e^{-t} \Longrightarrow dx = -e^{-t}dt\ \therefore \log x = -t$

x	0	1
t	∞	0

$$\therefore I = \int_{\infty}^{0} (e^{-t})^m (-t)^n (-e^{-t}) dt = \int_{0}^{\infty} (e^{-t})^m (-t)^n (e^{-t}) dt$$

$$I = (-1)^n \int_{0}^{\infty} e^{-(m+1)t} (t)^n dt = (-1)^n \frac{\Gamma(n+1)}{(m+1)^{n+1}}$$